HARMONIC MORPHISMS FROM THE GRASSMANNIANS AND THEIR NON-COMPACT DUALS

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ABSTRACT. In this paper we give a unified framework for the construction of complex valued harmonic morphisms from the real, complex and quaternionic Grassmannians and their non-compact duals. This gives a positive answer to the corresponding open existence problem in the real and quaternionic cases.

1. INTRODUCTION

The notion of a minimal submanifold of a given ambient space is of great importance in differential geometry. Harmonic morphisms $\phi : (M, g) \to (N, h)$ between Riemannian manifolds are useful tools for the construction of such objects, see Theorem 2.3 below. Harmonic morphisms are solutions to over-determined non-linear systems of partial differential equations determined by the geometric data of the manifolds involved. For this reason they can be difficult to find and have no general existence theory, not even locally. On the contrary, most metrics on a 3-dimensional domain M^3 do not allow any local harmonic morphisms with values in a surface N^2, see [3]. This makes it interesting to find geometric and topological conditions on the manifolds (M, g) and (N, h), ensuring the existence of such maps. For the general theory of harmonic morphisms between Riemannian manifolds, we refer to the excellent book [2] and the regularly updated on-line bibliography [6].

For the existence of harmonic morphisms $\phi : (M, g) \to (N, h)$ it is an advantage that the target manifold N is a surface, i.e. of dimension 2. In this case the problem is invariant under conformal changes of the metric on N^2. Therefore, at least for local studies, the codomain can be taken to be the standard complex plane.

It is known that, in several cases when the domain (M, g) is an irreducible Riemannian symmetric space, there exist complex valued solutions to the problem, see for example [7, 13]. This has led the authors to the following conjecture.

Conjecture 1.1. *Let (M^m, g) be an irreducible Riemannian symmetric space of dimension $m \geq 2$. For each point $p \in M$ there exists a complex valued harmonic morphism $\phi : U \to \mathbb{C}$ defined on an open neighbourhood U of p. If the space (M, g) is of non-compact type then the domain U can be chosen to be the whole of M.*

It is well-known that any holomorphic map from a Kähler manifold to a Riemann surface is a harmonic morphism. This means that the conjecture is true whenever the domain (M, g) is a Hermitian symmetric space; in particular a complex Grassmannian

$$\mathbf{SU}(p+q)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q))$$

or its non-compact dual

$$\mathbf{SU}(p, q)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q)),$$

which can be realized as a bounded symmetric domain in $\mathbb{C}^{pq}$.

In this paper we construct explicit complex valued harmonic morphisms defined *globally* on the non-compact irreducible Riemannian symmetric spaces

$$\mathbf{SO}_0(p, q)/\mathbf{SO}(p) \times \mathbf{SO}(q),$$

when $p \notin \{q, q \pm 1\}$, and

$$\mathbf{Sp}(p, q)/\mathbf{Sp}(p) \times \mathbf{Sp}(q)$$

when $p \neq q$. We prove the general duality Theorem 7.1 for complex valued harmonic morphisms from Riemannian symmetric spaces. This is then employed to yield locally defined solutions from the compact real Grassmannians

$$\mathbf{SO}(p+q)/\mathbf{SO}(p) \times \mathbf{SO}(q),$$

with $p \notin \{q, q \pm 1\}$, and the quaternionic Grassmannians

$$\mathbf{Sp}(p+q)/\mathbf{Sp}(p) \times \mathbf{Sp}(q)$$

when $p \neq q$.

Throughout this article we assume that all our objects such as manifolds, maps etc. are smooth, i.e. in the C^∞-category. For our notation concerning Lie groups we refer to the wonderful book [10].

2. Harmonic Morphisms

We are interested in complex valued harmonic morphisms from the real, complex and quaternionic Grassmannians and their non-compact duals. These are Riemannian manifolds, but our methods involve harmonic morphisms from the more general semi-Riemannian manifolds, see [12].

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Let M and N be two manifolds of dimensions m and n, respectively. Then a semi-Riemannian metric g on M gives rise to the notion of a Laplacian on (M, g) and real-valued harmonic functions $f : (M, g) \to \mathbb{R}$. This can be generalized to the concept of a harmonic map $\phi : (M, g) \to (N, h)$ between semi-Riemannian manifolds being a solution to a semi-linear system of partial differential equations, see [2].

Definition 2.1. A map $\phi : (M, g) \to (N, h)$ between semi-Riemannian manifolds is called a *harmonic morphism* if, for any harmonic function $f : U \to \mathbb{R}$ defined on an open subset U of N with $\phi^{-1}(U)$ non-empty, the composition $f \circ \phi : \phi^{-1}(U) \to \mathbb{R}$ is a harmonic function.

The following characterization of harmonic morphisms between semi-Riemannian manifolds is due to Fuglede, and generalizes the corresponding well-known result of [4, 9] in the Riemannian case. For the definition of horizontal conformality we refer to [2].

Theorem 2.2. [5] *A map $\phi : (M, g) \to (N, h)$ between semi-Riemannian manifolds is a harmonic morphism if and only if it is a horizontally (weakly) conformal harmonic map.*

The next result generalizes the corresponding well-known theorem of Baird and Eells in the Riemannian case, see [1]. It gives the theory of harmonic morphisms a strong geometric flavour and shows that the case when $n = 2$ is particularly interesting. In that case the conditions characterizing harmonic morphisms are then independent of conformal changes of the metric on the surface N^2. For the definition of horizontal homothety we refer to [2].

Theorem 2.3. [7] *Let $\phi : (M^m, g) \to (N^n, h)$ be a horizontally conformal submersion from a semi-Riemannian manifold (M^m, g) to a Riemannian manifold (N^n, h). If*

    i. *$n = 2$, then ϕ is harmonic if and only if ϕ has minimal fibres,*
    ii. *$n \geq 3$, then two of the following conditions imply the other:*
        (a) *ϕ is a harmonic map,*
        (b) *ϕ has minimal fibres,*
        (c) *ϕ is horizontally homothetic.*

In what follows we are mainly interested in complex valued functions $\phi, \psi : (M, g) \to \mathbb{C}$ from semi-Riemannian manifolds. In this situation the metric g induces the complex-valued Laplacian $\tau(\phi)$ and the gradient $\mathrm{grad}(\phi)$ with values in the complexified tangent bundle $T^{\mathbb{C}}M$ of M. We extend the metric g to be complex bilinear on $T^{\mathbb{C}}M$ and define the symmetric bilinear operator κ by

$$\kappa(\phi, \psi) = g(\mathrm{grad}(\phi), \mathrm{grad}(\psi)).$$

Two maps $\phi, \psi : M \to \mathbb{C}$ are said to be *orthogonal* if
$$\kappa(\phi, \psi) = 0.$$
The harmonicity and horizontal conformality of $\phi : (M, g) \to \mathbb{C}$ are given by the following relations
$$\tau(\phi) = 0 \quad \text{and} \quad \kappa(\phi, \phi) = 0.$$

Definition 2.4. Let (M, g) be a semi-Riemannian manifold. A set
$$\Omega = \{\phi_i : M \to \mathbb{C} \mid i \in I\}$$
of complex valued functions is said to be an *orthogonal harmonic family* on M if for all $\phi, \psi \in \Omega$
$$\tau(\phi) = 0 \quad \text{and} \quad \kappa(\phi, \psi) = 0.$$

Remark 2.5. For a finite orthogonal harmonic family $\{\phi_1, \ldots, \phi_k\}$ on a Riemannian manifold (M, g), the map
$$\Phi = (\phi_1, \ldots, \phi_k) : M \to \mathbb{C}^k$$
is a *pseudo horizontally (weakly) conformal* map. See for example Definition 8.2.3 and Example 8.2.6 of [2].

The next result shows that the elements of an orthogonal harmonic family can be used to produce a variety of harmonic morphisms. The main aim of this paper is to construct such families on the Riemannian symmetric spaces that we are dealing with.

Theorem 2.6. [7] *Let (M, g) be a semi-Riemannian manifold and*
$$\Omega = \{\phi_k : M \to \mathbb{C} \mid k = 1, \ldots, n\}$$
be a finite orthogonal harmonic family on (M, g). Let $\Phi : M \to \mathbb{C}^n$ be the map given by $\Phi = (\phi_1, \ldots, \phi_n)$ and U be an open subset of $\mathbb{C}^n$ containing the image $\Phi(M)$ of Φ. If
$$\tilde{\mathcal{F}} = \{F_i : U \to \mathbb{C} \mid i \in I\}$$
is a family of holomorphic functions then
$$\mathcal{F} = \{\psi : M \to \mathbb{C} \mid \psi = F(\phi_1, \ldots, \phi_n), \ F \in \tilde{\mathcal{F}}\}$$
is an orthogonal harmonic family on M.

Proof. The statement is a direct consequence of the fact that if
$$\psi_1 = F_1(\phi_1, \ldots, \phi_n), \quad \psi_2 = F_2(\phi_1, \ldots, \phi_n)$$
are elements of $\mathcal{F}$ then
$$\tau(\psi_r) = \sum_{k=1}^{n} \frac{\partial F_r}{\partial \phi_k} \tau(\phi_k) + \sum_{k,l=1}^{n} \frac{\partial^2 F_r}{\partial \phi_k \partial \phi_l} \kappa(\phi_k, \phi_l),$$

$$\kappa(\psi_r, \psi_s) \;=\; \sum_{k,l=1}^{n} \frac{\partial F_r}{\partial \phi_k} \frac{\partial F_s}{\partial \phi_l} \, \kappa(\phi_k, \phi_l).$$

$$\square$$

3. The Model Spaces

In this section we introduce models for some Riemannian symmetric spaces which are useful for our purposes. For more details, we refer to the classical works [11, 8]. Let $\mathbb{D}$ be one of the associative division algebras of the real numbers $\mathbb{R}$, complex numbers $\mathbb{C} = \{x + yi \mid x, y \in \mathbb{R}\}$ or the quaternions $\mathbb{H} = \{z + wj \mid z, w \in \mathbb{C}\}$ of real dimension $d = 1, 2, 4$, respectively. For the quaternions $\mathbb{H}$ we frequently make use of their standard representation in $\mathbb{C}^{2 \times 2}$ given by

$$z + wj \mapsto \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}.$$

By I_n we denote the $n \times n$ identity matrix and introduce the matrix

$$I_{pq} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}.$$

Let $\mathbb{D}^{(p+q) \times p}$ be the real vector space of $(p+q) \times p$ matrices, with entries in $\mathbb{D}$, equipped with either the semi-Euclidean inner product $(,)$ given by

$$(X, Y) = \mathfrak{Re}\{\mathrm{trace}(X^* I_{pq} Y)\} = \mathfrak{Re}\{-\sum_{k,l=1}^{p} \bar{x}_{kl} y_{kl} + \sum_{k=p+1}^{p+q} \sum_{l=1}^{p} \bar{x}_{kl} y_{kl}\}$$

or the Euclidean one $\langle , \rangle$ satisfying

$$\langle X, Y \rangle = \mathfrak{Re}\{\mathrm{trace}(X^* Y)\} = \mathfrak{Re}\{\sum_{k=1}^{p+q} \sum_{l=1}^{p} \bar{x}_{kl} y_{kl}\}.$$

Furthermore let $\mathbf{GL}_p(\mathbb{D})$ be the Lie group of the invertible $\mathbb{D}^{p \times p}$ matrices and $U_{pq}(\mathbb{D})$, $U_{pq}^*(\mathbb{D})$ be the open subsets of $\mathbb{D}^{(p+q) \times p}$ given by

$$U_{pq}(\mathbb{D}) = \{ \begin{pmatrix} X_0 \\ X_1 \end{pmatrix} \in \mathbb{D}^{(p+q) \times p} \mid -X_0^* X_0 + X_1^* X_1 < 0 \},$$

$$U_{pq}^*(\mathbb{D}) = \{ \begin{pmatrix} X_0 \\ X_1 \end{pmatrix} \in \mathbb{D}^{(p+q) \times p} \mid X_0^* X_0 + X_1^* X_1 \in \mathbf{GL}_p(\mathbb{D}) \}.$$

By the condition $-X_0^* X_0 + X_1^* X_1 < 0$ we mean that for each non-zero $x \in \mathbb{D}^p$ the real number

$$x^*(-X_0^* X_0 + X_1^* X_1)x = -(X_0 x)^*(X_0 x) + (X_1 x)^*(X_1 x)$$

is negative. The Lie group $\mathbf{GL}_p(\mathbb{D})$ acts on $U_{pq}(\mathbb{D})$ and $U_{pq}^*(\mathbb{D})$ by multiplication on the right and the quotient spaces

$$U_{pq}(\mathbb{D})/\mathbf{GL}_p(\mathbb{D}) \ \text{ and } \ U_{pq}^*(\mathbb{D})/\mathbf{GL}_p(\mathbb{D})$$

are differentiable manifolds of real dimension dpq.

Let $\Sigma_{pq}(\mathbb{D})$ and $\Sigma_{pq}^*(\mathbb{D})$ be the closed subsets of $\mathbb{D}^{(p+q)\times p}$ given by

$$\Sigma_{pq}(\mathbb{D}) = \{ \begin{pmatrix} X_0 \\ X_1 \end{pmatrix} \in \mathbb{D}^{(p+q)\times p} \mid -X_0^*X_0 + X_1^*X_1 = -I_p \},$$

$$\Sigma_{pq}^*(\mathbb{D}) = \{ \begin{pmatrix} X_0 \\ X_1 \end{pmatrix} \in \mathbb{D}^{(p+q)\times p} \mid X_0^*X_0 + X_1^*X_1 = I_p \}.$$

For $K_p(\mathbb{R}) = \mathbf{SO}(p)$, $K_p(\mathbb{C}) = \mathbf{SU}(p)$ and $K_p(\mathbb{H}) = \mathbf{Sp}(p)$ the Lie group $K_p(\mathbb{D})$ acts on $\Sigma_{pq}(\mathbb{D})$ and $\Sigma_{pq}^*(\mathbb{D})$ by multiplication on the right and the quotient spaces

$$\Sigma_{pq}(\mathbb{D})/K_p(\mathbb{D}) \ \text{ and } \ \Sigma_{pq}^*(\mathbb{D})/K_p(\mathbb{D})$$

can be identified with $U_{pq}(\mathbb{D})/\mathbf{GL}_p(\mathbb{D})$ and $U_{pq}^*(\mathbb{D})/\mathbf{GL}_p(\mathbb{D})$, respectively. The metrics $(,)$ and $\langle,\rangle$ on $\mathbb{D}^{(p+q)\times p}$ restricted to the closed subsets $\Sigma_{pq}(\mathbb{D})$ and $\Sigma_{pq}^*(\mathbb{D})$ induce uniquely determined Riemannian metrics on the quotient spaces $\Sigma_{pq}(\mathbb{D})/K_p(\mathbb{D})$ and $\Sigma_{pq}^*(\mathbb{D})/K_p(\mathbb{D})$, making the natural projections

$$\Sigma_{pq}(\mathbb{D}) \to \Sigma_{pq}(\mathbb{D})/K_p(\mathbb{D}) \ \text{ and } \ \Sigma_{pq}^*(\mathbb{D}) \to \Sigma_{pq}^*(\mathbb{D})/K_p(\mathbb{D})$$

into Riemannian submersions. The Riemannian manifolds obtained this way are the real, complex and quaternionic Grassmannians and their non-compact duals

$$
\begin{array}{ll}
\mathbf{SO}(p+q)/\mathbf{SO}(p)\times\mathbf{SO}(q) & \mathbf{SO}_0(p,q)/\mathbf{SO}(p)\times\mathbf{SO}(q) \\
\mathbf{SU}(p+q)/\mathbf{S}(\mathbf{U}(p)\times\mathbf{U}(q)) & \mathbf{SU}(p,q)/\mathbf{S}(\mathbf{U}(p)\times\mathbf{U}(q)) \\
\mathbf{Sp}(p+q)/\mathbf{Sp}(p)\times\mathbf{Sp}(q) & \mathbf{Sp}(p,q)/\mathbf{Sp}(p)\times\mathbf{Sp}(q).
\end{array}
$$

These are well known irreducible Riemannian symmetric spaces and the complex cases are distinguished by the fact that they carry a natural Kähler structure. The two spaces

$$\mathbf{SO}_0(p,q)/\mathbf{SO}(p)\times\mathbf{SO}(q) \ \text{ and } \ \mathbf{SO}_0(q,p)/\mathbf{SO}(q)\times\mathbf{SO}(p),$$

obtained from each other by interchanging p and q, are isomorphic. The same applies to the other pairs listed above.

Theorem 3.1. *Let the manifold $U_{pq}(\mathbb{D})$ be equipped with the semi-Euclidean metric $(,)$. Then the natural projection*

$$\pi : U_{pq}(\mathbb{D}) \to U_{pq}(\mathbb{D})/\mathbf{GL}_p(\mathbb{D})$$

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has the following property: if $\hat{\phi} : U_{pq}(\mathbb{D}) \to (N, h)$ is a $\mathbf{GL}_p(\mathbb{D})$-invariant harmonic morphism into a Riemannian manifold (N, h), then the induced map $\phi : U_{pq}(\mathbb{D})/\mathbf{GL}_p(\mathbb{D}) \to (N, h)$ is a harmonic morphism on the quotient space.

Theorem 3.2. *Let the manifold $U_{pq}^*(\mathbb{D})$ be equipped with the Euclidean metric $\langle, \rangle$. Then the natural projection*

$$\pi^* : U_{pq}^*(\mathbb{D}) \to U_{pq}^*(\mathbb{D})/\mathbf{GL}_p(\mathbb{D})$$

has the following property: if $\hat{\phi}^* : U_{pq}^*(\mathbb{D}) \to (N, h)$ is a $\mathbf{GL}_p(\mathbb{D})$-invariant harmonic morphism into a Riemannian manifold, then the induced map $\phi^* : U_{pq}^*(\mathbb{D})/\mathbf{GL}_p(\mathbb{D}) \to (N, h)$ is a harmonic morphism on the quotient space.

We shall now prove Theorem 3.1. The result of Theorem 3.2 can be proved in a similar way. This is left to the reader as an exercise.

Proof. For simplicity we introduce the notation

$$x_0 = \begin{pmatrix} I_p \\ 0 \end{pmatrix}.$$

Let G denote one of the groups $\mathbf{SO}_0(p, q)$, $\mathbf{SU}(p, q)$ or $\mathbf{Sp}(p, q)$, depending on the case at hand, and write

$$M = U_{pq}(\mathbb{D})/\mathbf{GL}_p(\mathbb{D}).$$

The group G acts isometrically on $U_{pq}(\mathbb{D})$, and as π is equivariant, it maps any fibre of π into a fibre of π. As the group is transitive on M, it is transitive on the fibres of π. By L_g we denote the multiplication from the left by the element $g \in G$ and use the same notation for this action on $U_{pq}(\mathbb{D})$, $\Sigma_{pq}(\mathbb{D})$ and on M. Let

$$\tilde{\pi} : \Sigma_{pq}(\mathbb{D}) \to M$$

denote the restriction of π to $\Sigma_{pq}(\mathbb{D})$.

The fibre over x_0 is the $\mathbf{GL}_p(\mathbb{D})$-orbit of x_0 in U_{pq}, which clearly is totally geodesic. From the isometric action of G, we conclude that all the fibres of π are totally geodesic.

Let $\hat{X}$ be a unit vector field around $\pi(x_0)$. As π is submersive, we can find a basic vector field X around x_0 such that $d\pi(X) = \hat{X}$. As $\hat{X} = d\pi(X)(x_0) = d\tilde{\pi}(X)(x_0)$, X_{x_0} is a unit vector. We have

$$
\begin{aligned}
\nabla d\pi(X, X)(x_0) &= \nabla_X d\pi(X)(x_0) - d\pi(\nabla_X X)(x_0) \\
&= (\nabla_{\hat{X}} \hat{X})(\pi(x_0)) - d\pi(\nabla_X X)(x_0) \\
&= (\nabla_{\hat{X}} \hat{X})(\tilde{\pi}(x_0)) - d\tilde{\pi}(\nabla_X X)(x_0)
\end{aligned}
$$

$$= \ 0,$$

as $\tilde{\pi}$ is a Riemannian submersion and X is also a basic vector field for $\tilde{\pi}$. If, on the other hand, V is a vertical vector field around x_0,

$$\nabla d\pi(V,V) = -d\pi(\nabla_V V) = 0,$$

as π has totally geodesic fibres. Since $\tau(\pi) = \text{trace}\,\nabla d\pi$, we have proved that $\tau(\pi)(x_0) = 0$. The group G is transitive on $\Sigma_{pq}(\mathbb{D})$ and L_g is an isometry, so we see that

$$\tau(\pi)(L_g(x_0)) = \tau(\pi \circ L_g)(x_0) = \tau(L_g \circ \pi)(x_0) = 0.$$

This means that $\tau(\pi)(x) = 0$ for all $x \in \Sigma_{pq}(\mathbb{D})$.

Let us now assume that $\hat{\phi} = \phi \circ \pi$ is a $\mathbf{GL}_p(\mathbb{D})$-invariant harmonic morphism on some open subset of $U_{pq}(\mathbb{D})$. Without loss of generality we can assume that x_0 is contained in the domain of $\hat{\phi}$. As $d\pi$ maps its horizontal space isometrically at x_0 onto the tangent space of M, it is clear that ϕ is horizontally conformal at $\pi(x_0)$. By translating with elements in G, we conclude that ϕ is horizontally conformal everywhere. To prove that ϕ is harmonic, note that

$$
\begin{aligned}
0 &= \tau(\hat{\phi})(x_0) \\
&= d\phi(\tau(\pi)(x_0)) + \text{trace}\,\nabla d\phi_{\pi(x_0)}(d\pi_{x_0}, d\pi_{x_0}) \\
&= \tau(\phi)(\pi(x_0)).
\end{aligned}
$$

Once again, by translating with elements in G, we see that ϕ is harmonic. $\qquad\square$

4. The non-compact complex cases

In this section we give a simple description of how to construct orthogonal harmonic families on the non-compact irreducible Hermitian symmetric spaces

$$\mathbf{SU}(p,q)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q)) = U_{pq}(\mathbb{C})/\mathbf{GL}_p(\mathbb{C}).$$

We employ Theorem 3.1 and lift the problem into the subset $U_{pq}(\mathbb{C})$ of $\mathbb{C}^{(p+q)\times p}$ equipped with the semi-Euclidean metric $(,)$. This gives the following expressions for the operators τ and κ

$$
\tau(\phi) \ = \ -4 \sum_{k,l=1}^{p} \frac{\partial^2 \phi}{\partial z_{kl} \partial \bar{z}_{kl}} + 4 \sum_{k=p+1}^{p+q} \sum_{l=1}^{p} \frac{\partial^2 \phi}{\partial z_{kl} \partial \bar{z}_{kl}},
$$

$$
\kappa(\phi,\psi) \ = \ -2 \sum_{k,l=1}^{p} \Big(\frac{\partial \phi}{\partial z_{kl}} \frac{\partial \psi}{\partial \bar{z}_{kl}} + \frac{\partial \phi}{\partial \bar{z}_{kl}} \frac{\partial \psi}{\partial z_{kl}} \Big)
$$

$$+2\sum_{k=p+1}^{p+q}\sum_{l=1}^{p}\Big(\frac{\partial\phi}{\partial z_{kl}}\frac{\partial\psi}{\partial\bar{z}_{kl}}+\frac{\partial\phi}{\partial\bar{z}_{kl}}\frac{\partial\psi}{\partial z_{kl}}\Big).$$

Proposition 4.1. *Let* $\Phi:U_{pq}(\mathbb{C})\to\mathbb{C}^{q\times p}$ *be the map given by*

$$\Phi:\begin{pmatrix}Z_0\\Z_1\end{pmatrix}\mapsto Z_1\cdot Z_0^{-1}.$$

Then the complex valued components of Φ *form an orthogonal harmonic family of* $\mathbf{GL}_p(\mathbb{C})$*-invariant functions on* $U_{pq}(\mathbb{C})$.

Proof. This is a direct consequence of the formulae for τ and κ given above and the fact that Φ is holomorphic. $\square$

The components of the map Φ are $\mathbf{GL}_p(\mathbb{C})$-invariant, and so induce holomorphic functions on $U_{pq}(\mathbb{C})/\mathbf{GL}_p(\mathbb{C})$ which constitute an orthogonal harmonic family on that space.

5. The non-compact real cases

We shall now introduce a method for constructing orthogonal harmonic families on the non-compact irreducible Riemannian symmetric spaces

$$\mathbf{SO}_0(p,q)/\mathbf{SO}(p)\times\mathbf{SO}(q)=U_{pq}(\mathbb{R})/\mathbf{GL}_p(\mathbb{R}),$$

when $p\notin\{q,q\pm1\}$. It is easily seen that our method does not work in the special cases of $p\in\{q,q\pm1\}$. As in the complex case, we employ Theorem 3.1 and lift the problem into the set $U_{pq}(\mathbb{R})$.

Let p,r be positive integers, $s=p+2r$ and on $\mathbb{R}^{(p+p+r+r)\times p}$ introduce the coordinates

$$\begin{pmatrix}A\\B\\W\\\bar{W}\end{pmatrix}=\begin{pmatrix}X_0-X_1\\X_0+X_1\\X_2+iX_3\\X_2-iX_3\end{pmatrix}\quad\text{where}\quad\begin{pmatrix}X_0\\X_1\\X_2\\X_3\end{pmatrix}\in\mathbb{R}^{(p+p+r+r)\times p}.$$

The semi-Euclidean metric $(,)$ on $\mathbb{R}^{(p+s)\times p}$ gives the following expressions for the operators τ and κ

$$\tau(\phi)\;=\;-4\sum_{k,l=1}^{p}\frac{\partial^2\phi}{\partial a_{kl}\partial b_{kl}}+4\sum_{k=1}^{r}\sum_{l=1}^{p}\frac{\partial^2\phi}{\partial w_{kl}\partial\bar{w}_{kl}},$$

$$\kappa(\phi,\psi)\;=\;-2\sum_{k,l=1}^{p}\Big(\frac{\partial\phi}{\partial a_{kl}}\frac{\partial\psi}{\partial b_{kl}}+\frac{\partial\phi}{\partial b_{kl}}\frac{\partial\psi}{\partial a_{kl}}\Big)$$

$$+2\sum_{k=1}^{r}\sum_{l=1}^{p}\Big(\frac{\partial\phi}{\partial w_{kl}}\frac{\partial\psi}{\partial\bar{w}_{kl}}+\frac{\partial\phi}{\partial\bar{w}_{kl}}\frac{\partial\psi}{\partial w_{kl}}\Big).$$

Proposition 5.1. *Let $\hat{M}$ be an element of the complexification $\mathfrak{so}(p,r)^{\mathbb{C}}$ of the Lie algebra $\mathfrak{so}(p,r)$ and define the map $\hat{\Phi} : \mathbb{R}^{(p+s)\times p} \to \mathbb{C}^{(p+r)\times p}$ by*

$$\hat{\Phi} : X \mapsto \begin{pmatrix} A \\ W \end{pmatrix} + \hat{M} \cdot \begin{pmatrix} B \\ \bar{W} \end{pmatrix}.$$

Then the complex valued components of $\hat{\Phi}$ constitute an orthogonal harmonic family on $\mathbb{R}^{(p+s)\times p}$.

Proof. This is a simple calculation using the above formulae for the operators τ and κ. $\qquad\square$

The following result generalizes the construction of complex valued harmonic morphisms from the odd-dimensional real hyperbolic spaces

$$\mathbb{R}H^{2r-1} = \mathbf{SO}_0(1, 2r-1)/\mathbf{SO}(1) \times \mathbf{SO}(2r-1)$$

presented in [7].

Proposition 5.2. *Let $\Phi : U_{ps}(\mathbb{R}) \to \mathbb{C}^{r\times p}$ be the map given by*

$$\Phi : X \mapsto W \cdot A^{-1}.$$

Then the complex valued components of the map Φ form an orthogonal harmonic family of $\mathbf{GL}_p(\mathbb{R})$-invariant functions on $U_{ps}(\mathbb{R})$.

Proof. This is a direct consequence of Proposition 5.1 in the case when $\hat{M} = 0$. $\qquad\square$

The next result generalizes a special case of the construction of complex valued harmonic morphisms from the even-dimensional real hyperbolic spaces

$$\mathbb{R}H^{2r-2} = \mathbf{SO}_0(1, 2r-2)/\mathbf{SO}(1) \times \mathbf{SO}(2r-2)$$

presented in [13].

Proposition 5.3. *For an integer $r \geq 2$ and $M \in \mathfrak{so}(r,\mathbb{C})$ let the map $\hat{\Phi} : U_{ps}(\mathbb{R}) \to \mathbb{C}^{(r-1)\times p}$ be defined by*

$$\hat{\Phi} : X \mapsto S \cdot (W + M \cdot \bar{W}) \cdot A^{-1}$$

where S is the matrix given by

$$S = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & m_{1r} \\ 0 & 1 & \cdots & 0 & 0 & m_{2r} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & m_{r-2,r} \\ 0 & 0 & \cdots & 0 & 1 & m_{r-1,r} \end{pmatrix}.$$

Then $\hat{\Phi}$ is independent of the last row of the matrix X thus inducing a map $\Phi : U_{p,s-1}(\mathbb{R}) \to \mathbb{C}^{(r-1)\times p}$. The complex valued components of Φ constitute an orthogonal harmonic family of $\mathbf{GL}_p(\mathbb{R})$-invariant functions on $U_{p,s-1}(\mathbb{R})$.

Proof. Let $t = 2p + 2r$ index the last row of X. If $k = 1, \ldots, p$ then

$$2\frac{\partial\hat{\Phi}}{\partial x_{tk}} \cdot A$$

$$= S(iE_{rk} - M \cdot iE_{rk})$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & m_{1r} \\ 0 & 1 & \cdots & 0 & 0 & m_{2r} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & m_{r-2,r} \\ 0 & 0 & \cdots & 0 & 1 & m_{r-1,r} \end{pmatrix} \begin{pmatrix} 0 & \cdots & 0 & -im_{1r} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & -im_{r-1,r} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & i & 0 & \cdots & 0 \end{pmatrix}$$

$$= 0.$$

The rest follows by Proposition 5.1. $\qquad\square$

6. THE NON-COMPACT QUATERNIONIC CASES

In this section we construct orthogonal harmonic families on the non-compact irreducible Riemannian symmetric spaces

$$\mathbf{Sp}(p,q)/\mathbf{Sp}(p) \times \mathbf{Sp}(q) = U_{pq}(\mathbb{H})/\mathbf{GL}_p(\mathbb{H}),$$

with $p \neq q$. Our method does not work in the special cases of $p = q$.

Let p, r be positive integers and set $q = p + r$. For elements

$$Q = \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \end{pmatrix} \in U_{pq}(\mathbb{H})$$

we shall use the complex notation

$$Q_0 = Z + Wj, \quad Q_1 = X + Yj, \quad Q_2 = U + Vj, \quad P = Q_0 - Q_1$$

and the standard representation of $\mathbb{H}^{n\times n}$ in $\mathbb{C}^{2n\times 2n}$:

$$A + Bj \mapsto \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}.$$

Lemma 6.1. *If p, r are positive integers and $q = p + r$ then*

$$\Phi : U_{pq}(\mathbb{H}) \to \mathbb{H}^{r\times p}, \quad \Phi : Q \mapsto Q_2(Q_0 - Q_1)^{-1}$$

is a $\mathbf{GL}_p(\mathbb{H})$-invariant harmonic map on $U_{pq}(\mathbb{H})$.

Proof. Using the complex representation of Φ,

$$\Phi : Q \mapsto \begin{pmatrix} U & V \\ -\bar{V} & \bar{U} \end{pmatrix} \begin{pmatrix} Z - X & W - Y \\ \bar{Y} - \bar{W} & \bar{Z} - \bar{X} \end{pmatrix}^{-1},$$

we see that the tension field $\tau(\Phi)$ is given by the following expression

$$\begin{aligned}
\tau(\Phi) &= -4 \sum_{k,l=1}^{p} \left[\frac{\partial^2 \Phi}{\partial z_{kl} \partial \bar{z}_{kl}} + \frac{\partial^2 \Phi}{\partial w_{kl} \partial \bar{w}_{kl}} \right] \\
&\quad +4 \sum_{k,l=1}^{p} \left[\frac{\partial^2 \Phi}{\partial x_{kl} \partial \bar{x}_{kl}} + \frac{\partial^2 \Phi}{\partial y_{kl} \partial \bar{y}_{kl}} \right] \\
&\quad +4 \sum_{k=1}^{r} \sum_{l=1}^{p} \left[\frac{\partial^2 \Phi}{\partial u_{kl} \partial \bar{u}_{kl}} + \frac{\partial^2 \Phi}{\partial v_{kl} \partial \bar{v}_{kl}} \right].
\end{aligned}$$

It is obvious that the last sum vanishes. Using the following properties for derivations D_1, D_2 of the matrix algebra $\mathbb{C}^{2p \times 2p}$ and an invertible element z thereof,

$$\begin{aligned}
D_1(z^{-1}) &= -z^{-1} D_1(z) z^{-1}, \\
D_2(D_1(z^{-1})) &= z^{-1}(D_2(z)z^{-1}D_1(z) - D_2(D_1(z)) \\
&\quad\quad\quad\quad\quad\quad + D_1(z)z^{-1}D_2(z))z^{-1},
\end{aligned}$$

we see that

$$\begin{aligned}
\tau(\Phi) &= -4Q_2 P^{-1} \Bigg(\sum_{k,l=1}^{p} \Big[\frac{\partial P}{\partial z_{kl}} P^{-1} \frac{\partial P}{\partial \bar{z}_{kl}} + \frac{\partial P}{\partial \bar{z}_{kl}} P^{-1} \frac{\partial P}{\partial z_{kl}} \\
&\quad\quad\quad + \frac{\partial P}{\partial w_{kl}} P^{-1} \frac{\partial P}{\partial \bar{w}_{kl}} + \frac{\partial P}{\partial \bar{w}_{kl}} P^{-1} \frac{\partial P}{\partial w_{kl}} \Big] \\
&\quad\quad - \sum_{k,l=1}^{p} \Big[\frac{\partial P}{\partial x_{kl}} P^{-1} \frac{\partial P}{\partial \bar{x}_{kl}} + \frac{\partial P}{\partial \bar{x}_{kl}} P^{-1} \frac{\partial P}{\partial x_{kl}} \\
&\quad\quad\quad + \frac{\partial P}{\partial y_{kl}} P^{-1} \frac{\partial P}{\partial \bar{y}_{kl}} + \frac{\partial P}{\partial \bar{y}_{kl}} P^{-1} \frac{\partial P}{\partial y_{kl}} \Big] \Bigg) P^{-1} \\
&= -4Q_2 P^{-1} \Bigg(\sum_{k,l=1}^{p} \Big[\begin{pmatrix} E_{kl} & 0 \\ 0 & 0 \end{pmatrix} P^{-1} \begin{pmatrix} 0 & 0 \\ 0 & E_{kl} \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 0 \\ 0 & E_{kl} \end{pmatrix} P^{-1} \begin{pmatrix} E_{kl} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & E_{kl} \\ 0 & 0 \end{pmatrix} P^{-1} \begin{pmatrix} 0 & 0 \\ -E_{kl} & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 0 \\ -E_{kl} & 0 \end{pmatrix} P^{-1} \begin{pmatrix} 0 & E_{kl} \\ 0 & 0 \end{pmatrix} \Big]
\end{aligned}$$

$$- \sum_{k,l=1}^{p} \left[\begin{pmatrix} -E_{kl} & 0 \\ 0 & 0 \end{pmatrix} P^{-1} \begin{pmatrix} 0 & 0 \\ 0 & -E_{kl} \end{pmatrix} \right.$$

$$+ \begin{pmatrix} 0 & 0 \\ 0 & -E_{kl} \end{pmatrix} P^{-1} \begin{pmatrix} -E_{kl} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -E_{kl} \\ 0 & 0 \end{pmatrix} P^{-1} \begin{pmatrix} 0 & 0 \\ E_{kl} & 0 \end{pmatrix}$$

$$\left. + \begin{pmatrix} 0 & 0 \\ E_{kl} & 0 \end{pmatrix} P^{-1} \begin{pmatrix} 0 & -E_{kl} \\ 0 & 0 \end{pmatrix} \right])$$

$$= \quad 0.$$

This shows that Φ is harmonic. $\qquad\square$

The next result generalizes the construction of complex valued harmonic morphisms from the quaternionic hyperbolic spaces

$$\mathbb{H}H^q = \mathbf{Sp}(1,q)/\mathbf{Sp}(1) \times \mathbf{SO}(q)$$

presented in [7].

Proposition 6.2. *Let* p, r *be positive integers,* $q = p + r$ *and* $\Phi :$ $U_{pq}(\mathbb{H}) \to \mathbb{C}^{r \times 2p}$ *the map given by*

$$\Phi : Q \mapsto \begin{pmatrix} U & V \end{pmatrix} \begin{pmatrix} Z - X & W - Y \\ \bar{Y} - \bar{W} & \bar{Z} - \bar{X} \end{pmatrix}^{-1}.$$

Then the complex valued components of Φ *form an orthogonal harmonic family of* $\mathbf{GL}_p(\mathbb{H})$*-invariant functions on* $U_{pq}(\mathbb{H})$.

Proof. It follows from Lemma 6.1 that the components of Φ are harmonic. The operator κ is given by the following equation:

$$\kappa(\phi, \psi)$$

$$= \quad -2 \sum_{k,l=1}^{p} \left(\frac{\partial \phi}{\partial z_{kl}} \frac{\partial \psi}{\partial \bar{z}_{kl}} + \frac{\partial \phi}{\partial w_{kl}} \frac{\partial \psi}{\partial \bar{w}_{kl}} + \frac{\partial \psi}{\partial z_{kl}} \frac{\partial \phi}{\partial \bar{z}_{kl}} + \frac{\partial \psi}{\partial w_{kl}} \frac{\partial \phi}{\partial \bar{w}_{kl}} \right)$$

$$+ 2 \sum_{k,l=1}^{p} \left(\frac{\partial \phi}{\partial x_{kl}} \frac{\partial \psi}{\partial \bar{x}_{kl}} + \frac{\partial \phi}{\partial y_{kl}} \frac{\partial \psi}{\partial \bar{y}_{kl}} + \frac{\partial \psi}{\partial x_{kl}} \frac{\partial \phi}{\partial \bar{x}_{kl}} + \frac{\partial \psi}{\partial y_{kl}} \frac{\partial \phi}{\partial \bar{y}_{kl}} \right)$$

$$+ 2 \sum_{k=1}^{r} \sum_{l=1}^{p} \left(\frac{\partial \phi}{\partial u_{kl}} \frac{\partial \psi}{\partial \bar{u}_{kl}} + \frac{\partial \phi}{\partial v_{kl}} \frac{\partial \psi}{\partial \bar{v}_{kl}} + \frac{\partial \psi}{\partial u_{kl}} \frac{\partial \phi}{\partial \bar{u}_{kl}} + \frac{\partial \psi}{\partial v_{kl}} \frac{\partial \phi}{\partial \bar{v}_{kl}} \right).$$

It is a direct consequence of the definition of Φ that

$$\frac{\partial \phi_{ij}}{\partial \bar{u}_{kl}} = 0, \quad \frac{\partial \phi_{ij}}{\partial \bar{v}_{kl}} = 0.$$

Applying the relation $D(P^{-1}) = -P^{-1} D(P) P^{-1}$ we easily see that

$$\frac{\partial \phi_{ij}}{\partial z_{kl}} \frac{\partial \phi_{rs}}{\partial \bar{z}_{kl}} = \frac{\partial \phi_{ij}}{\partial x_{kl}} \frac{\partial \phi_{rs}}{\partial \bar{x}_{kl}}, \quad \frac{\partial \phi_{ij}}{\partial w_{kl}} \frac{\partial \phi_{rs}}{\partial \bar{w}_{kl}} = \frac{\partial \phi_{ij}}{\partial y_{kl}} \frac{\partial \phi_{rs}}{\partial \bar{y}_{kl}}.$$

This shows that the $\mathbf{GL}_p(\mathbb{H})$-invariant components of Φ form an orthogonal harmonic family on $U_{pq}(\mathbb{H})$. $\qquad\square$

7. THE DUALITY

In this section we show how a locally defined complex valued harmonic morphism from a Riemannian symmetric space G/K of non-compact type gives rise to a dual locally defined harmonic morphism from its compact dual U/K, and vice versa. Recall that any harmonic morphism between real analytic Riemannian manifolds is real analytic, see [2].

Let W be an open subset of G/K and $\phi : W \to \mathbb{C}$ a real analytic map. By composing ϕ with the projection $G \to G/K$ we obtain a real analytic K-invariant map $\hat{\phi} : \hat{W} \to \mathbb{C}$ from some open subset $\hat{W}$ of G. Let $G^{\mathbb{C}}$ denote the complexification of the Lie group G. Then $\hat{\phi}$ extends uniquely to a K-invariant holomorphic map $\phi^{\mathbb{C}} : W^{\mathbb{C}} \to \mathbb{C}$ from some open subset $W^{\mathbb{C}}$ of $G^{\mathbb{C}}$. By restricting this map to $U \cap W^{\mathbb{C}}$ and factoring through the projection $U \to U/K$, we obtain a real analytic map $\phi^* : W^* \to \mathbb{C}$ from some open subset W^* of U/K.

Theorem 7.1. *Let $\mathcal{F}$ be a family of maps $\phi : W \to \mathbb{C}$ and $\mathcal{F}^*$ be the dual family consisting of the maps $\phi^* : W^* \to \mathbb{C}$ constructed as above. Then $\mathcal{F}^*$ is an orthogonal harmonic family if and only if $\mathcal{F}$ is an orthogonal harmonic family.*

Proof. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of the Lie algebra of G, where $\mathfrak{k}$ is the Lie algebra of K. Furthermore let the left-invariant vector fields $X_1, \ldots, X_n \in \mathfrak{p}$ form a global orthonormal frame for the distribution generated by $\mathfrak{p}$.

Let $\hat{\phi}$ be the lift of $\phi : W \to \mathbb{C}$, via the natural projection $\pi : G \to G/K$, defined on the open subset $\pi^{-1}(W)$ of G. We shall now assume that ϕ is a harmonic morphism, i.e.

$$\tau(\hat{\phi}) = \sum_{k=1}^n X_k^2(\hat{\phi}) = 0, \quad \kappa(\hat{\phi}, \hat{\phi}) = \sum_{k=1}^n X_k(\hat{\phi})^2 = 0.$$

By construction and by the unique continuation property of real analytic functions, the extension $\hat{\phi}^{\mathbb{C}}$ of $\hat{\phi}$ satisfies the same equations.

The Lie algebra of U has the decomposition $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$ and the left-invariant vector fields $i\bar{X}_1, \ldots, i\bar{X}_n \in i\mathfrak{p}$ form a global orthonormal frame for the distribution generated by $i\mathfrak{p}$. Let $\hat{\phi}^*$ be the lift of $\phi^* : W^* \to \mathbb{C}$, via the natural projection $\pi^* : U \to U/K$, defined on the

open subset $(\pi^*)^{-1}(W^*)$ of U. Then

$$\tau(\hat{\phi}^*) = \sum_{k=1}^{n}(iX_k)^2(\hat{\phi}^*) = -\sum_{k=1}^{n}X_k^2(\hat{\phi}^{\mathbb{C}}) = 0,$$

$$\kappa(\hat{\phi}^*, \hat{\phi}^*) = \sum_{k=1}^{n}(iX_k)(\hat{\phi}^*)^2 = -\sum_{k=1}^{n}X_k(\hat{\phi}^{\mathbb{C}})^2 = 0.$$

This shows that ϕ^* is a harmonic morphism.

Let us now assume that $\mathcal{F}$ is an orthogonal harmonic family and that $\phi, \psi \in \mathcal{F}$. Then according to Theorem 2.6 the sum $\phi + \psi$ is a harmonic morphism. Hence the dual maps ϕ^*, ψ^* and $(\phi + \psi)^*$ are harmonic morphisms and we have

$$\kappa((\phi + \psi)^*, (\phi + \psi)^*) = \kappa(\phi^*, \phi^*) + 2\kappa(\phi^*, \psi^*) + \kappa(\psi^*, \psi^*) = 0.$$

Then the relations $\kappa(\phi^*, \phi^*) = \kappa(\psi^*, \psi^*) = 0$ imply that $\kappa(\phi^*, \psi^*) = 0$ in other words the harmonic dual maps ϕ^* and ψ^* are orthogonal. This shows that $\mathcal{F}^*$ is an orthogonal harmonic family. The converse is similar. $\qquad\square$

We shall apply Theorem 7.1 to construct orthogonal harmonic families on the compact real, complex and hyperbolic Grassmannians. For that purpose we now explicitly describe the duality in the real and hyperbolic cases.

Example 7.2. For the special orthogonal group $\mathbf{SO}(p + q)$ we have the following Cartan decomposition

$$\mathfrak{so}(p + q) = (\mathfrak{so}(p) \oplus \mathfrak{so}(q)) \oplus \mathfrak{p},$$

where

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & X \\ -X^t & 0 \end{pmatrix} \;\middle|\; X \in \mathbb{R}^{p \times q} \right\}.$$

Let G be the connected subgroup of $\mathbf{SO}(p + q, \mathbb{C})$ with Lie algebra

$$\mathfrak{g} = (\mathfrak{so}(p) \oplus \mathfrak{so}(q)) \oplus i\mathfrak{p}.$$

It is easy to see that G is the identity component of the group

$$\{g \in \mathbf{SO}(p + q, \mathbb{C}) \mid g^* I_{pq} g = I_{pq}\}.$$

Introduce the matrix

$$\eta = \begin{pmatrix} -iI_p & 0 \\ 0 & I_q \end{pmatrix}.$$

The map $\rho : G \to \mathbf{SO}_0(p, q)$ given by

$$\rho(g) = \eta g \bar{\eta}$$

is an isomorphism, and we have the following commutative diagram:

$$\mathbf{SO}(p+q,\mathbb{C}) \supset G \xrightarrow[\cong]{\rho} \mathbf{SO}_0(p,q) \longrightarrow \mathbf{SO}_0(p,q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$$

$$\Sigma_{pq}(\mathbb{R}) \xrightarrow{\ \pi\ } \Sigma_{pq}(\mathbb{R})/\mathbf{SO}(p).$$

Let $\phi : W \to \mathbb{C}$ be a locally defined, real analytic and $\mathbf{SO}(p) \times \mathbf{SO}(q)$-invariant map on $\mathbf{SO}_0(p,q)$. Then the composition $\phi \circ \rho$ is a $\mathbf{SO}(p) \times \mathbf{SO}(q)$-invariant map defined locally on G. This we can extend to a unique holomorphic map $\phi^{\mathbb{C}} : W^{\mathbb{C}} \to \mathbb{C}$ on some open subset $W^{\mathbb{C}}$ of $\mathbf{SO}(p+q,\mathbb{C})$. The restriction ϕ^* of $\phi^{\mathbb{C}}$ to $W^* = W^{\mathbb{C}} \cap \mathbf{SO}(p+q)$ is a locally defined $\mathbf{SO}(p) \times \mathbf{SO}(q)$-invariant on $\mathbf{SO}(p+q)$. The maps ϕ, ϕ^* induce maps locally defined on $\Sigma_{pq}(\mathbb{R})$ and $\Sigma_{pq}^*(\mathbb{R})$, respectively, which we also denote by ϕ, ϕ^*. Untangling the definitions, we see that

$$\phi^*\left(\begin{pmatrix} X \\ Y \end{pmatrix}\right) = \phi\left(\begin{pmatrix} X \\ iY \end{pmatrix}\right),$$

where $X \in \mathbb{R}^{p\times p}$ and $Y \in \mathbb{R}^{q\times p}$. By employing Theorem 7.1, we see that ϕ^* is a harmonic morphism if and only if ϕ is.

Example 7.3. For positive integers p, q and $n = p + q$ we introduce the matrices

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \ I_{pq} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}, \ K = \begin{pmatrix} I_{pq} & 0 \\ 0 & I_{pq} \end{pmatrix}, \ \eta = \begin{pmatrix} I_n & 0 \\ 0 & I_{pq} \end{pmatrix}.$$

Then it is easily seen that

$$\eta J \eta = KJ, \quad \eta K \eta = K.$$

Recall the following definitions:

$$\begin{aligned}
\mathbf{Sp}(n,\mathbb{C}) &= \{g \in \mathbf{SL}_{2n}(\mathbb{C}) \mid g^t Jg = J\} \\
\mathbf{Sp}(n) &= \mathbf{Sp}(n,\mathbb{C}) \cap \mathbf{U}(2n) = \{g \in \mathbf{SU}(2n) \mid gJ = J\bar{g}\} \\
\mathbf{Sp}(p,q) &= \{g \in \mathbf{SL}_{2n}(\mathbb{C}) \mid gJ = J\bar{g}, \ g^* Kg = K\}.
\end{aligned}$$

Introduce the subgroup

$$G = \{g \in \mathbf{Sp}(n,\mathbb{C}) \mid g^* Kg = K\}$$

of $\mathbf{Sp}(n,\mathbb{C})$. It is easy to see that $\mathbf{Sp}(p) \times \mathbf{Sp}(q)$ is contained in G and that the map

$$\rho : G \to \mathbf{Sp}(p,q), \quad g \mapsto \eta g \eta$$

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establishes an isomorphism between G and $\mathbf{Sp}(p,q)$ which preserves $\mathbf{Sp}(p) \times \mathbf{Sp}(q)$. As in the real case we have a commutative diagram

$$\mathbf{Sp}(n,\mathbb{C}) \supset G \xrightarrow[\cong]{\rho} \mathbf{Sp}(p,q) \longrightarrow \mathbf{Sp}(p,q)/\mathbf{Sp}(p) \times \mathbf{Sp}(q)$$

$$\Sigma_{pq}(\mathbb{H}) \xrightarrow{\pi} \Sigma_{pq}(\mathbb{H})/\mathbf{Sp}(p).$$

Assume that $\phi : W \to \mathbb{C}$ is a locally defined $\mathbf{Sp}(p) \times \mathbf{Sp}(q)$-invariant harmonic morphism on $\mathbf{Sp}(p,q)$ and let $\phi^* : W^* \to \mathbb{C}$ be the $\mathbf{Sp}(p) \times \mathbf{Sp}(q)$-invariant harmonic morphism locally defined on $\mathbf{Sp}(p+q)$, induced by $\phi \circ \rho$. Applying arguments similar to those we have used in the real case yield the relation

$$\phi^*\left(\begin{pmatrix} Z & W \\ X & Y \\ U & V \\ -\bar{W} & \bar{Z} \\ -\bar{Y} & \bar{X} \\ -\bar{V} & \bar{U} \end{pmatrix}\right) = \phi\left(\begin{pmatrix} Z & -W \\ X & -Y \\ U & -V \\ \bar{W} & \bar{Z} \\ -\bar{Y} & -\bar{X} \\ -\bar{V} & -\bar{U} \end{pmatrix}\right).$$

8. THE COMPACT COMPLEX CASES

In this section we give a simple description of how to construct orthogonal harmonic families on open subsets of the complex Grassmannians

$$\mathbf{SU}(p+q)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q)) = U^*_{pq}(\mathbb{C})/\mathbf{GL}_p(\mathbb{C}).$$

The Euclidean metric $\langle, \rangle$ on $\mathbb{C}^{(p+q)\times p}$ gives the following expressions for the operators τ and κ:

$$\tau(\phi) = 4\sum_{k=1}^{p+q}\sum_{l=1}^{p} \frac{\partial^2 \phi}{\partial z_{kl} \partial \bar{z}_{kl}},$$

$$\kappa(\phi, \psi) = 2\sum_{k=1}^{p+q}\sum_{l=1}^{p}\left(\frac{\partial \phi}{\partial z_{kl}}\frac{\partial \psi}{\partial \bar{z}_{kl}} + \frac{\partial \phi}{\partial \bar{z}_{kl}}\frac{\partial \psi}{\partial z_{kl}}\right).$$

Let the set $V^*_{pq}(\mathbb{C})$ be defined by

$$V^*_{pq}(\mathbb{C}) = \{\begin{pmatrix} Z_0 \\ Z_1 \end{pmatrix} \in U^*_{pq}(\mathbb{C}) \mid \det Z_0 \neq 0\}.$$

Proposition 8.1. *Let* $\Phi^* : V^*_{pq}(\mathbb{C}) \to \mathbb{C}^{q\times p}$ *be the map given by*

$$\Phi^* : \begin{pmatrix} Z_0 \\ Z_1 \end{pmatrix} \mapsto Z_1 \cdot Z_0^{-1}.$$

Then the complex valued components of Φ^* constitute an orthogonal harmonic family of $\mathbf{GL}_p(\mathbb{C})$-invariant functions on $V_{pq}^*(\mathbb{C})$.

Proof. This is a direct consequence of the fact that Φ^* is holomorphic and the formulae for τ and κ given above. $\qquad\square$

9. THE COMPACT REAL CASES

We shall now introduce a method for constructing orthogonal harmonic families on open subsets of the real Grassmannians

$$\mathbf{SO}(p+q)/\mathbf{SO}(p) \times \mathbf{SO}(q) = U_{pq}^*(\mathbb{R})/\mathbf{GL}_p(\mathbb{R}),$$

with $p \notin \{q, q \pm 1\}$. Let p, r be positive integers, $s = p + 2r$ and introduce the complex coordinates

$$\begin{pmatrix} Z \\ \bar{Z} \\ W \\ \bar{W} \end{pmatrix} = \begin{pmatrix} X_0 + iX_1 \\ X_0 - iX_1 \\ X_2 + iX_3 \\ X_2 - iX_3 \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix} \in \mathbb{R}^{(p+p+r+r) \times p}.$$

The Euclidean metric $\langle,\rangle$ on $\mathbb{R}^{(p+s) \times p}$ gives the following equalities for the operators τ and κ

$$\tau(\phi) = 4 \sum_{k,l=1}^p \frac{\partial^2 \phi}{\partial z_{kl} \partial \bar{z}_{kl}} + 4 \sum_{k=1}^s \sum_{l=1}^p \frac{\partial^2 \phi}{\partial w_{kl} \partial \bar{w}_{kl}},$$

$$\kappa(\phi, \psi) = 2 \sum_{k,l=1}^p \left(\frac{\partial \phi}{\partial z_{kl}} \frac{\partial \psi}{\partial \bar{z}_{kl}} + \frac{\partial \phi}{\partial \bar{z}_{kl}} \frac{\partial \psi}{\partial z_{kl}} \right)$$

$$+ 2 \sum_{k=1}^s \sum_{l=1}^p \left(\frac{\partial \phi}{\partial w_{kl}} \frac{\partial \psi}{\partial \bar{w}_{kl}} + \frac{\partial \phi}{\partial \bar{w}_{kl}} \frac{\partial \psi}{\partial w_{kl}} \right).$$

Proposition 9.1. *Let the matrix $\hat{M}$ be an element of the complex Lie algebra $\mathfrak{so}(p+r, \mathbb{C})$ and define the map $\hat{\Phi}^* : \mathbb{R}^{(p+s) \times p} \to \mathbb{C}^{(p+r) \times p}$ by*

$$\hat{\Phi}^* : X \mapsto \begin{pmatrix} Z \\ W \end{pmatrix} + \hat{M} \cdot \begin{pmatrix} \bar{Z} \\ \bar{W} \end{pmatrix}.$$

Then the complex valued components of $\hat{\Phi}^*$ constitute an orthogonal harmonic family on $\mathbb{R}^{(p+s) \times p}$.

Proof. This is a simple calculation using the above formulae for the operators τ and κ. $\qquad\square$

The following result generalizes the construction of complex valued harmonic morphisms from the odd-dimensional real projective spaces

$$\mathbb{R}P^{2r-1} = \mathbf{SO}(2r)/\mathbf{SO}(1) \times \mathbf{SO}(2r-1)$$

presented in [7]. For this we define the set $V_{pq}^*(\mathbb{R})$ by the formula

$$V_{pq}^*(\mathbb{R}) = \{X \in U_{pq}^*(\mathbb{R}) \mid \det Z \neq 0\}.$$

Proposition 9.2. *Let $\Phi^* : V_{ps}^*(\mathbb{R}) \to \mathbb{C}^{r\times p}$ be the map defined by*

$$\Phi^* : X \mapsto W \cdot Z^{-1}.$$

Then the complex valued components of Φ^* form an orthogonal harmonic family of $\mathbf{GL}_p(\mathbb{R})$-invariant functions on $V_{ps}^*(\mathbb{R})$.

Proof. This is a direct consequence of Proposition 9.1 in the case when $\hat{M} = 0$. $\square$

Proposition 9.3. *For an integer $r \geq 2$ and $M \in \mathfrak{so}(r, \mathbb{C})$ let the map $\hat{\Phi}^* : V_{ps}^*(\mathbb{R}) \to \mathbb{C}^{(r-1)\times p}$ be defined by*

$$\hat{\Phi}^* : X \mapsto S \cdot (W + M \cdot \bar{W}) \cdot Z^{-1}$$

where S is the matrix given by

$$S = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & m_{1r} \\ 0 & 1 & \cdots & 0 & 0 & m_{2r} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & m_{r-2,r} \\ 0 & 0 & \cdots & 0 & 1 & m_{r-1,r} \end{pmatrix}.$$

Then $\hat{\Phi}^*$ is independent of the last row of the matrix X, thus inducing a map $\Phi^* : V_{p,s-1}^*(\mathbb{R}) \to \mathbb{C}^{(r-1)\times p}$. The complex valued components of Φ^* form an orthogonal harmonic family of $\mathbf{GL}_p(\mathbb{R})$-invariant functions on $V_{p,s-1}^*(\mathbb{R})$.

Proof. Let $t = 2p + 2r$ index the last row of X. If $k = 1, \ldots, p$ then

$$2\frac{\partial \hat{\Phi}^*}{\partial x_{tk}} \cdot Z$$
$$= S(iE_{rk} - M \cdot iE_{rk})$$
$$= \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & m_{1r} \\ 0 & 1 & \cdots & 0 & 0 & m_{2r} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & m_{r-2,r} \\ 0 & 0 & \cdots & 0 & 1 & m_{r-1,r} \end{pmatrix} \begin{pmatrix} 0 & \cdots & 0 & -im_{1r} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & -im_{r-1,r} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & i & 0 & \cdots & 0 \end{pmatrix}$$
$$= 0.$$

The rest follows by Proposition 9.1. $\square$

10. The compact quaternionic cases

In this section we construct orthogonal harmonic families on open subsets of the quaternionic Grassmannians

$$\mathbf{Sp}(p+q)/\mathbf{Sp}(p) \times \mathbf{Sp}(q) = U_{pq}^*(\mathbb{H})/\mathbf{GL}_p(\mathbb{H}),$$

with $p \neq q$.

Let p, r be positive integers and set $q = p + r$. As before, we use the notation

$$\begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} Z + Wj \\ X + Yj \\ U + Vj \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \end{pmatrix} \in U_{pq}^*(\mathbb{H}).$$

Define the set

$$V_{pq}^*(\mathbb{H}) = \{Q \in U_{pq}^*(\mathbb{H}) \mid \det \begin{pmatrix} Z - X & Y - W \\ \bar{Y} + \bar{W} & \bar{Z} + \bar{X} \end{pmatrix} \neq 0\}.$$

Proposition 10.1. *Let p, r be positive integers, $q = p + r$ and let $\Phi^* : V_{pq}^*(\mathbb{H}) \to \mathbb{C}^{2r \times 2p}$ be the map given by*

$$\Phi^* : Q \mapsto \begin{pmatrix} U & -V \end{pmatrix} \begin{pmatrix} Z - X & Y - W \\ \bar{Y} + \bar{W} & \bar{Z} + \bar{X} \end{pmatrix}^{-1}.$$

Then the complex valued components of Φ^* constitute an orthogonal harmonic family of $\mathbf{GL}_p(\mathbb{H})$-invariant functions on $V_{pq}^*(\mathbb{H})$.

Proof. The statement follows from Theorem 7.1 combined with Proposition 6.2 and Example 7.3. □

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